



Division of Strength of Materials and Structures  
Faculty of Power and Aeronautical Engineering



# Finite element method (FEM1)

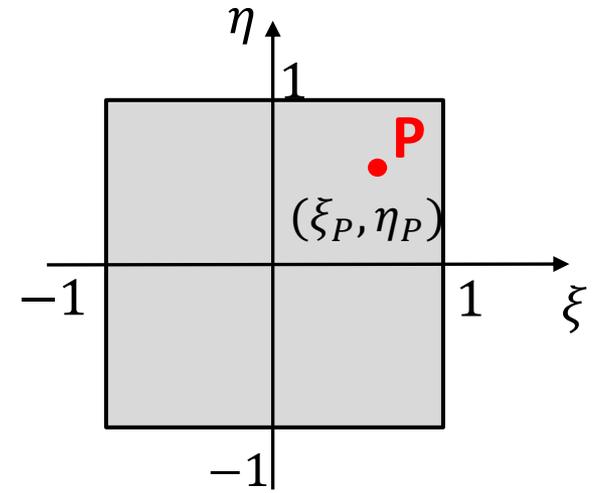
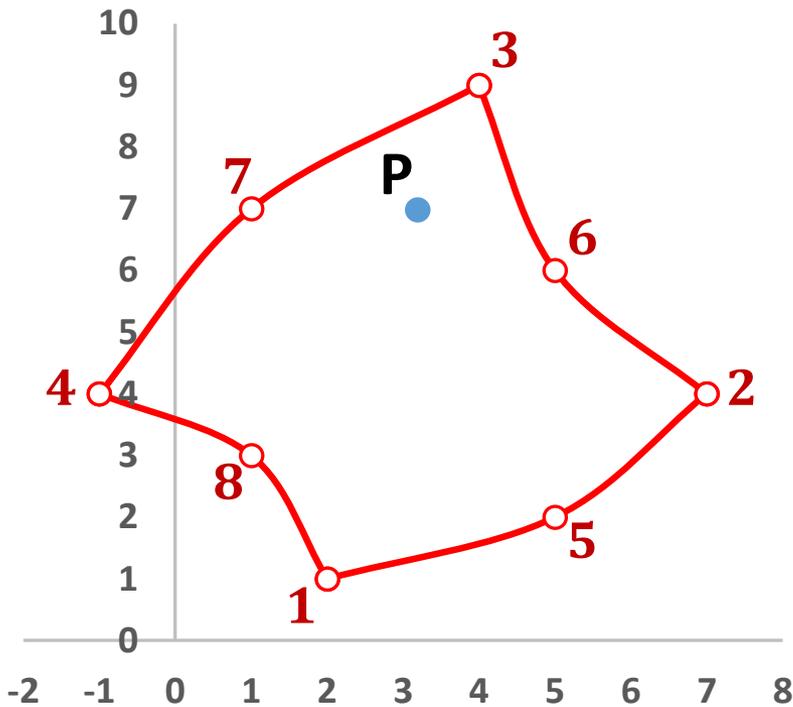
Lecture 4B. QUAD-8node element - example

03.2026

# Example 1. QUAD-8node: Find coordinates and det [J] at point P.

coordinates in the natural system:

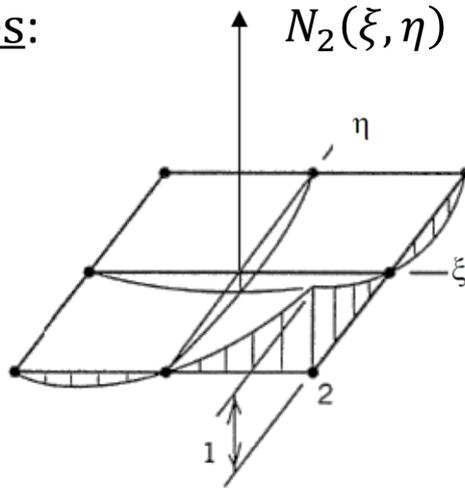
$$\xi_P = \eta_P = \frac{1}{\sqrt{3}}$$



$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix}_{8 \times 1} = \begin{Bmatrix} 2 \\ 7 \\ 4 \\ -1 \\ 5 \\ 5 \\ 1 \\ 1 \end{Bmatrix} ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix}_{8 \times 1} = \begin{Bmatrix} 4 \\ 9 \\ 4 \\ 2 \\ 6 \\ 7 \\ 3 \end{Bmatrix}$$

## Shape functions of an 8-node quadrilateral element

corner nodes:



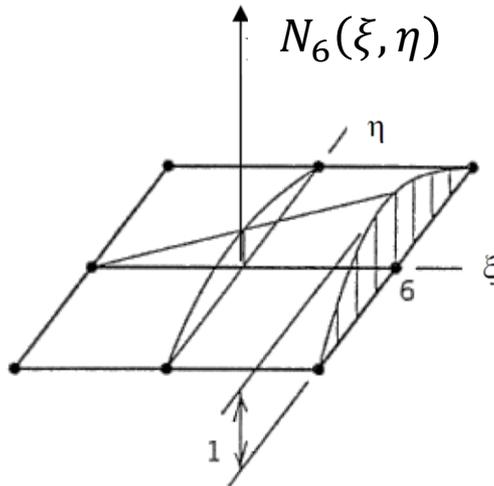
$$N_1(\xi, \eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi, \eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi, \eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi, \eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

midside nodes:



$$N_5(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6(\xi, \eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8(\xi, \eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

### Example shape function and its derivative for node 1

$$N_1(\xi, \eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \frac{\partial [(1-\xi)(1+\xi+\eta)]}{\partial \xi} =$$

$$= -\frac{1}{4}(1-\eta) \left[ \frac{\partial (1-\xi)}{\partial \xi} (1+\xi+\eta) + \frac{\partial (1+\xi+\eta)}{\partial \xi} (1-\xi) \right] =$$

$$= -\frac{1}{4}(1-\eta) \left[ -1 \cdot (1+\xi+\eta) + 1 \cdot (1-\xi) \right] =$$

$$= -\frac{1}{4}(1-\eta) [-1-\xi-\eta+1-\xi] = -\frac{1}{4}(1-\eta) [-2\xi-\eta] =$$

$$= \frac{1}{4}(1-\eta)(2\xi+\eta)$$

## Shape functions and their derivatives for an 8-node element

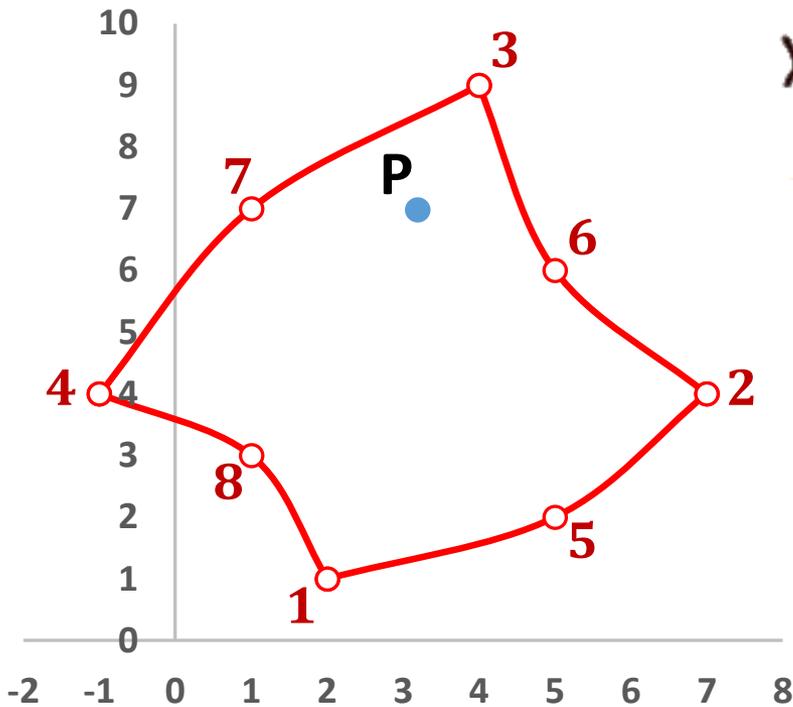
$i$	$\frac{\partial N_i}{\partial \xi}$	$\frac{\partial N_i}{\partial \eta}$
1	$\frac{1}{4}(1-\eta)(2\xi+\eta)$	$\frac{1}{4}(1-\xi)(\xi+2\eta)$
2	$\frac{1}{4}(1-\eta)(2\xi-\eta)$	$\frac{1}{4}(1+\xi)(2\eta-\xi)$
3	$\frac{1}{4}(1+\eta)(2\xi+\eta)$	$\frac{1}{4}(1+\xi)(\xi+2\eta)$
4	$\frac{1}{4}(1+\eta)(2\xi-\eta)$	$\frac{1}{4}(1-\xi)(2\eta-\xi)$
5	$-(1-\eta)\xi$	$-\frac{1}{2}(1-\xi^2)$
6	$\frac{1}{2}(1-\eta^2)$	$-(1+\xi)\cdot\eta$
7	$-(1+\eta)\xi$	$\frac{1}{2}(1-\xi^2)$
8	$-\frac{1}{2}(1-\eta^2)$	$-(1-\xi)\cdot\eta$

# Coordinates of point P approximated by shape functions

Cartesian coordinates:

$$x_p = \underbrace{[N(\xi_p, \eta_p)]}_{4 \times 8} \cdot \underbrace{\{x_i\}_e}_{8 \times 1} = 3.1925 \text{ mm}$$

$$y_p = \underbrace{[N(\xi_p, \eta_p)]}_{4 \times 8} \cdot \underbrace{\{y_i\}_e}_{8 \times 1} = 6.9761 \text{ mm}$$



$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 7 \\ 4 \\ -1 \\ 5 \\ 5 \\ 1 \\ 1 \end{Bmatrix} ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 9 \\ 4 \\ 2 \\ 6 \\ 7 \\ 3 \end{Bmatrix}$$

## Determinant of the Jacobian matrix for point P

$$\det[J] = \left( \frac{\partial[N(\xi,\eta)]}{\partial\xi} \{x_i\}_e \right)_{1 \times 8} \left( \frac{\partial[N(\xi,\eta)]}{\partial\eta} \{y_i\}_e \right)_{8 \times 1} - \left( \frac{\partial[N(\xi,\eta)]}{\partial\xi} \{y_i\}_e \right)_{1 \times 8} \left( \frac{\partial[N(\xi,\eta)]}{\partial\eta} \{x_i\}_e \right)_{8 \times 1}$$

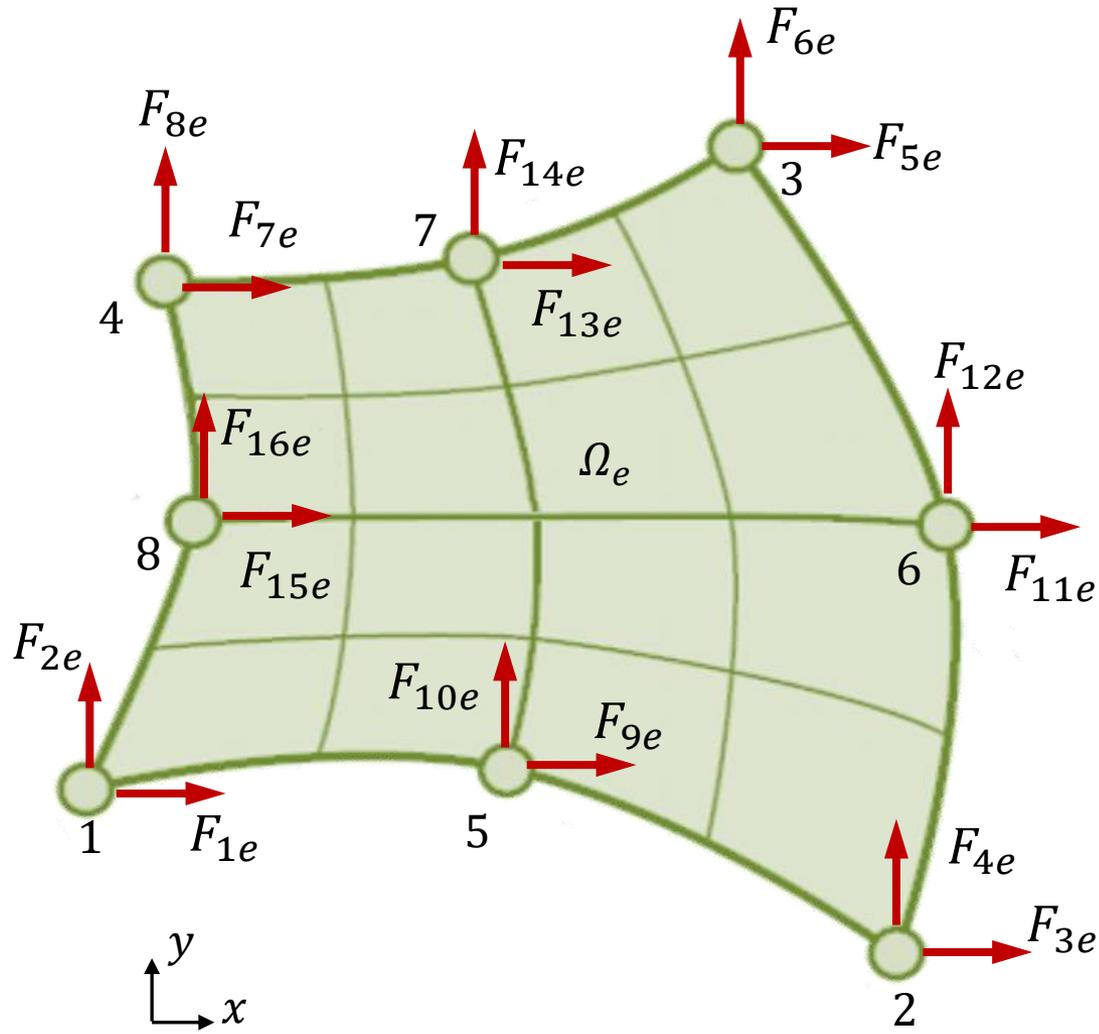
$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 7 \\ 4 \\ -1 \\ 5 \\ 5 \\ 1 \\ 1 \end{Bmatrix} ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 9 \\ 4 \\ 2 \\ 6 \\ 7 \\ 3 \end{Bmatrix}$$

$\xi$	0.57735							
$\eta$	0.57735							
i	1	2	3	4	5	6	7	8
N	-0.09623	-0.16667	0.096225	-0.16667	0.140883	0.525783	0.525783	0.140883
dN/d $\xi$	0.183013	0.061004	0.683013	0.227671	-0.24402	0.333333	-0.91068	-0.33333
dN/d $\eta$	0.183013	0.227671	0.683013	0.061004	-0.33333	-0.91068	0.333333	-0.24402
det[J]	<b>9.821367</b>							

# Equivalent load vector in the 8-node quadrilateral element

$$[F]_e$$

$16 \times 1$

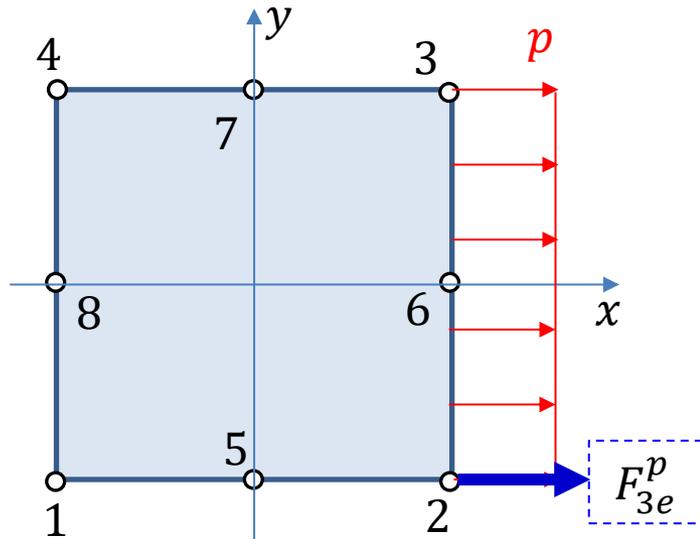


$$\{F\}_e = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ F_{16} \end{Bmatrix}_e$$

➔ Determination of the equivalent force in an 8-node element

## Example 2a Determination of the equivalent force in an 8-node element due to surface load

We will look for the equivalent force at node 2 resulting from the constant load  $p$  acting on the edge 2-3



equivalent load vector due to surface load :

$$[F^p]_e = t_e \int_0^l [p][N] ds$$

The work of the equivalent force  $F_{3e}^p$  on displacement 1

$$F_{3e}^p \cdot 1 = t_e \int_0^l p(x, y) u(\frac{l}{2}, y) dy$$

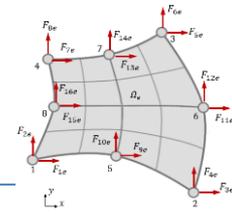
The work of load  $p(x, y)$  on displacement  $u(x, y)$

$$F_{3e}^p = t_e \int_0^l p N_2 dy$$

$$N_2(\xi, \eta) = -\frac{1}{4}(1 + \xi)(1 - \eta)(1 - \xi + \eta)$$

$$N_2(1, \eta) = -\frac{1}{2}(1 - \eta)\eta$$

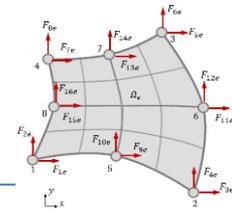
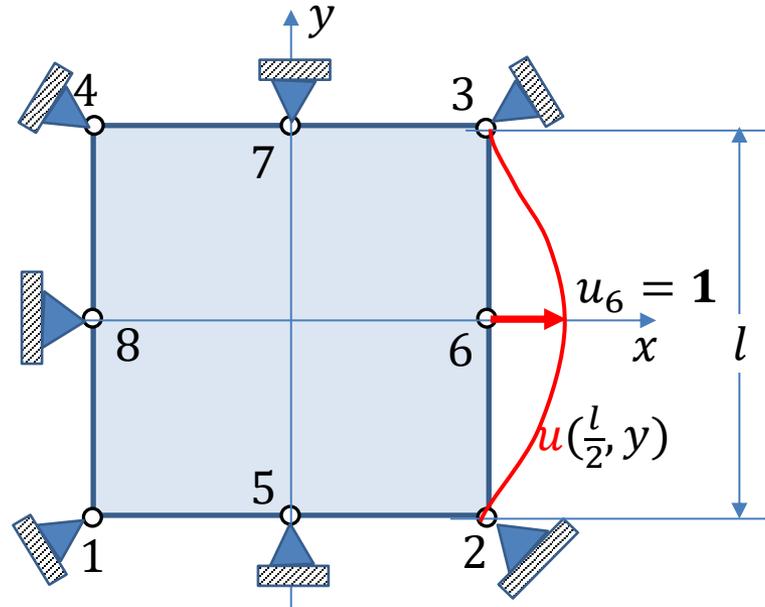
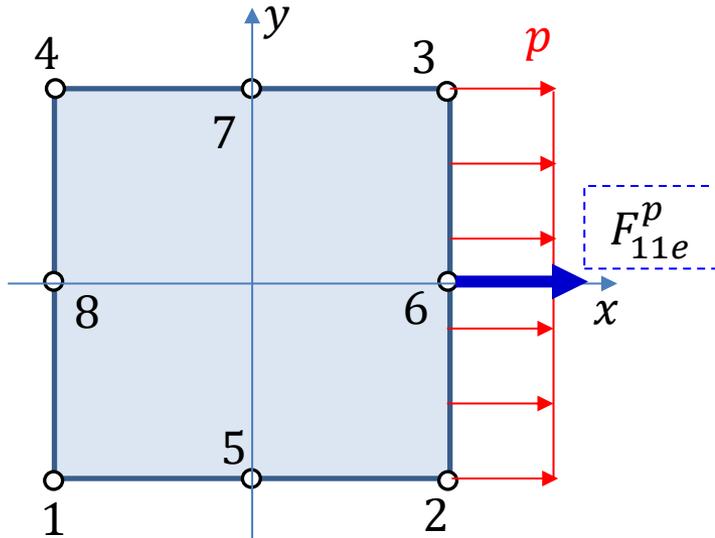
$$F_{3e}^p = t_e p \int_{-1}^1 \frac{1}{2}(\eta - 1)\eta \frac{l}{2} d\eta = \frac{pl}{4} t_e \left( \frac{1}{3}\eta^3 - \frac{1}{2}\eta^2 \right) \Big|_{-1}^1 = \frac{pl}{6} t_e$$



$$dy = \frac{l}{2} d\eta$$

## Example 2b Determination of the equivalent force in an 8-node element due to surface load

We will look for the equivalent force at node **6** resulting from the constant load  $p$  acting on the edge 2-3



equivalent load vector due to surface load :

$$[F^p]_e = t_e \int_0^l [p][N] ds$$

The work of the equivalent force  $F_{11e}^p$  on displacement 1

$$F_{11e}^p \cdot 1 = t_e \int_0^l p(x, y) u(\frac{l}{2}, y) dy$$

The work of load  $p(x, y)$  on displacement  $u(x, y)$

$$F_{11e}^p = t_e \int_0^l p N_6 dy$$

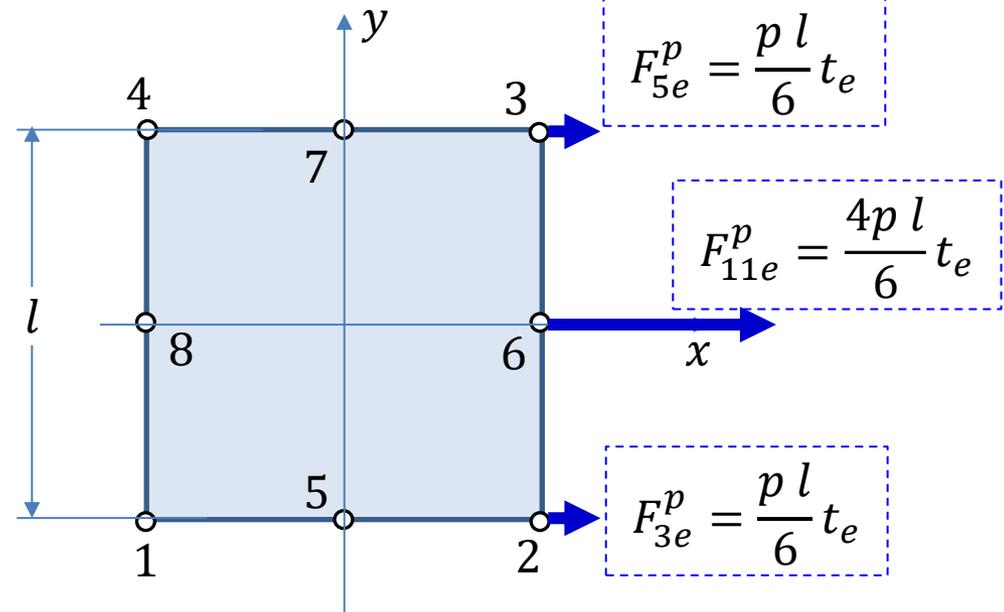
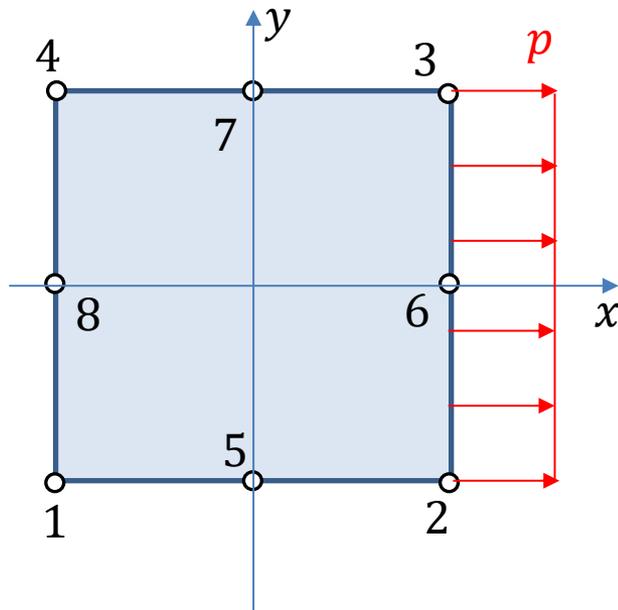
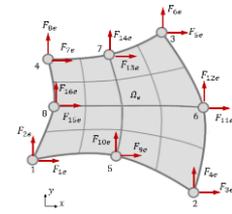
$$N_6(\xi, \eta) = \frac{1}{2}(1 + \xi)(1 - \eta^2)$$

$$N_6(1, \eta) = 1 - \eta^2$$

$$F_{11e}^p = t_e p \int_{-1}^1 (1 - \eta^2) \frac{l}{2} d\eta = \frac{pl}{2} t_e \left( \eta - \frac{1}{3} \eta^3 \right) \Big|_{-1}^1 = \frac{2pl}{3} t_e$$

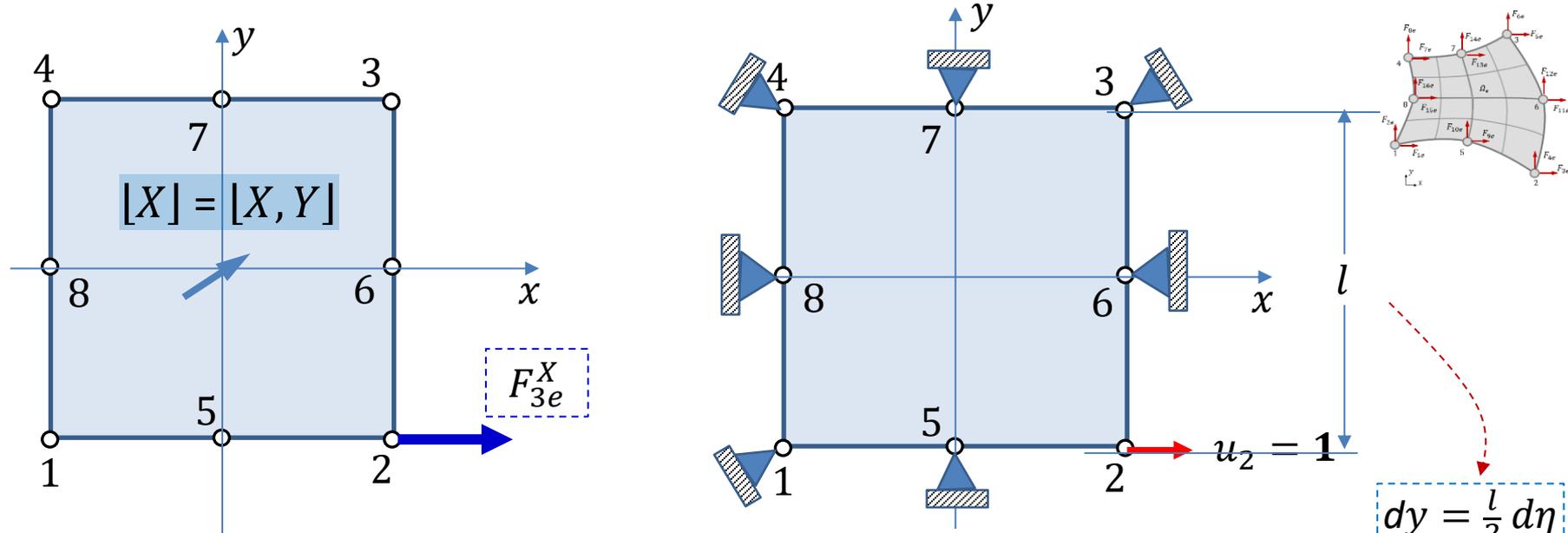
## Example 2 Determination of the equivalent force in an 8-node element due to surface load

Equivalent forces resulting from the constant load  $p$  acting on the edge 2-3



### Example 3 Determination of the equivalent force in an 8-node element due to mass loads

We will look for the equivalent force at node **2** resulting from the constant mass load **X**



equivalent load vector due to mass load:

$$[F^X]_e = t_e \int_{A_e} [X][N] dA_e$$

The work of the equivalent force  $F_{3e}^X$  acting on displacement 1

The work of the mass load  $X$  acting on displacement  $u(x, y)$

$$F_{3e}^X \cdot 1 = t_e \int_{A_e} X u(x, y) dA$$

$$F_{3e}^X = t_e \int_{A_e} X N_2 dA$$

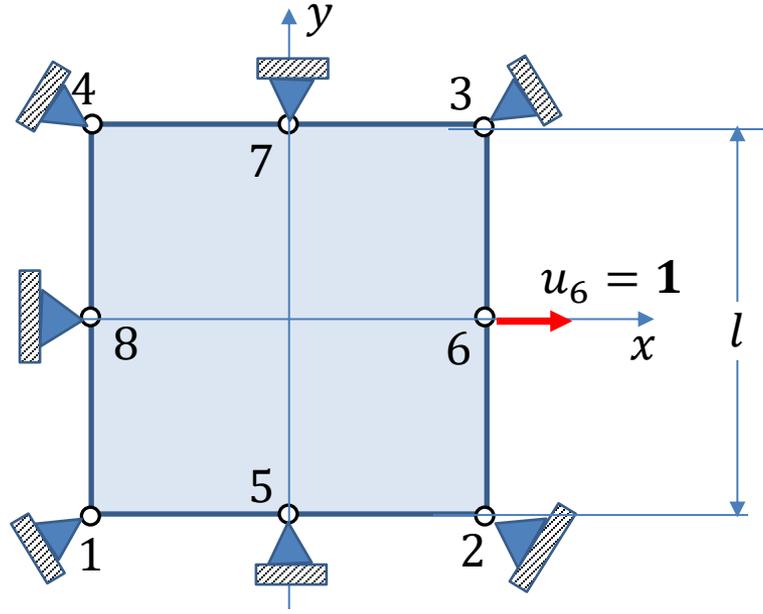
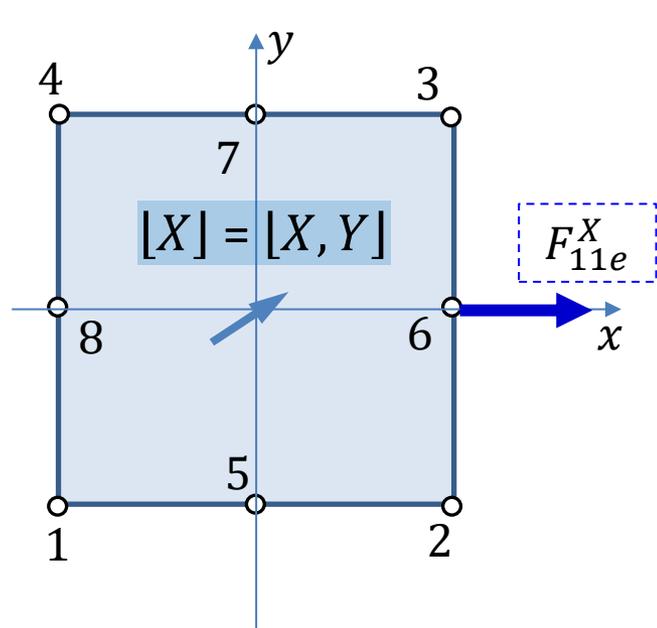
$$\begin{aligned} dy &= \frac{l}{2} d\eta \\ dx &= \frac{l}{2} d\xi \end{aligned}$$

$$N_2(\xi, \eta) = -\frac{1}{4}(1 + \xi)(1 - \eta)(1 - \xi + \eta)$$

$$F_{3e}^X = t_e X \int_{-1}^1 \int_{-1}^1 -\frac{1}{4}(1 + \xi)(1 - \eta)(1 - \xi + \eta) \frac{l}{2} d\xi \frac{l}{2} d\eta = -\frac{1}{12} t_e X l^2$$

### Example 3 Determination of the equivalent force in an 8-node element due to mass loads

We will look for the equivalent force at node 6 resulting from the constant mass load  $X$



equivalent load vector due to mass load:

$$[F^X]_e = t_e \int_{A_e} [X][N] dA_e$$

The work of the equivalent force  $F_{11e}^X$  acting on displacement 1

The work of the mass load  $X$  acting on displacement  $u(x, y)$

$$F_{11e}^X \cdot 1 = t_e \int_{A_e} X u(x, y) dA$$

$$F_{11e}^X = t_e \int_{A_e} X N_6 dA$$

$$\begin{aligned} dy &= \frac{l}{2} d\eta \\ dx &= \frac{l}{2} d\xi \end{aligned}$$

$$N_6(\xi, \eta) = \frac{1}{2}(1 + \xi)(1 - \eta^2)$$

$$F_{11e}^X = t_e X \int_{-1}^1 \int_{-1}^1 \frac{1}{2}(1 + \xi)(1 - \eta^2) \frac{l}{2} d\xi \frac{l}{2} d\eta = \frac{1}{3} t_e X l^2$$

### Example 3 Determination of the equivalent forces in an 8-node element due to mass loads

Equivalent forces resulting from the constant mass load  $\mathbf{X} = [X, Y]$

